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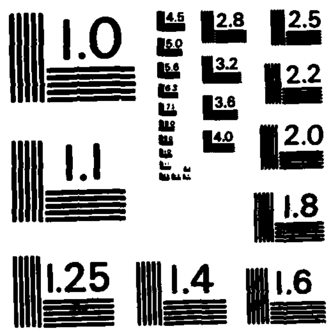

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EDF STATISTICS FOR TESTING FOR THE GAMMA DISTRIBUTION, WITH APPLICATIONS

By

A.N. Pettitt

and

M.A. Stephens

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EDF Statistics For Testing For The Gamma

Distribution, With Applications.

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1. INTRODUCTION

Several years ago the authors found percentage points for EDF statistics for the Gamma distribution with known shape parameter but unknown scale parameter (the origin of the distribution was assumed to be zero). These were issued as two Technical Reports (Pettitt, 1975, ~~Stephens~~, 1975). The gamma or equivalently the chi-squared distribution often occurs as a possible model for observations or as the distribution of some derived statistics. For example, in lifetime or survival studies the gamma distribution can be proposed as the distribution of lifetime or some function of lifetime. The chi-squared distribution occurs as the distribution for sums of squares in ANOVA tables and also in time series analysis, in the study of periodograms, and in multivariate analysis, as the distribution of squared radii. A review of possible uses of the gamma distribution is given by Johnson and Kotz (1970, §17).

Dispite the many uses of the gamma distribution, the application of the tables might seem to be limited, because the shape parameter must be known. In the earlier reports, ~~we gave~~ ARE an illustration of the use of EDF statistics to test that the sample variances of cells of an ANOVA table, with the same number of observations in each cell, all came from the same distribution. Since then, other applications have come to light and these will be presented in this report; also the test procedures and the tables will be reproduced to make the report complete in itself.

2. TEST PROCEDURES.

2.1. The Gamma Distribution.

The null hypothesis to be tested is

$$H_0\text{: a random sample } x_1,\ldots,x_n ,$$

with distribution function $F(x)$, comes from the gamma population with known shape parameter m but unknown scale parameter θ. The null hypothesis is therefore

$$H_0\colon F(x) = G(\theta x,m) , \tag{1}$$

where $G(z,m)$ is the gamma distribution function given by

$$G(z,m) = \int_0^z \frac{t^{m-1} e^{-t}}{\Gamma(m)}\, dt .$$

When the parameter θ is known then the null hypothesis H_0 can be tested using a whole variety of statistics, since the probability integral transformation, $u_i = G(x_i,m)$, produces $u_1,\ldots,u_n$ which behave like a random sample from the uniform $(0,1)$ distribution when H_0 is true. Pearson and Hartley (1972, p117) consider some suitable statistics.

When the parameter θ is unspecified and has to be estimated from the sample then the problem of testing H_0 becomes more complicated. Statistics which are both useful in practice and analytically tractable are EDF statistics, based on the empirical distribution function. In particular the weighted Cramér-von Mises statistics have been found to have good power properties.

A Cramér-von Mises type statistic is defined in general terms by

$$w_n^2 = n \int_{-\infty}^{\infty} \{F_n(x) - F_0(x;\theta)\}^2 \, \Psi\{F_0(x;\underset{\sim}{\theta})\} \, dF_0(x;\underset{\sim}{\theta}) \ ,$$

where $F_0(x;\underset{\sim}{\theta})$ is the distribution function specified by the goodness-of-fit null hypothesis, and in this particular case $F_0(x;\underset{\sim}{\theta}) = G(\theta x, m)$. The function $\Psi(\cdot)$ is a positive weight function. Other EDF statistics, such as the Kolmogorov-Smirnov type statistics tend to be less powerful than the Cramér-von Mises type statistics (see, for example, Stephens, 1974, §5). In this report we give the asymptotic distributions of three well known statistics, W^2, the Cramér-von Mises statistic, A^2, the Anderson-Darling statistic, and U^2, the Watson statistic, when θ is replaced by its maximum likelihood estimate $\hat{\theta} = m/\bar{x}$, where $\bar{x} = \Sigma x_i/n$.

For increasing sample size n, percentage points of the test statistics converge rapidly to the asymptotic percentage points. A very small modification to the value of the test statistic then makes it possible to use only the asymptotic points when making the test. The modifications are on the lines of those given in Stephens (1974) and in Pearson and Hartley (1972, Table 54).

2.2. Test Procedure.

For a set of observations x, we give here the procedure to be followed, to test H_0 given in (1).

(a) Put the observations in ascending order, $x_1 \le x_2 \le \cdots \le x_n$; let $\bar{x} = \Sigma x_i/n$.

(b) Use $\hat{\theta} = m/\bar{x}$ to calculate $z_i = \hat{\theta} x_i$, $i = 1,\ldots,n$.

(c) Find $w_i = G(z_i, m)$ for $i = 1,\ldots,n$, and the mean $\bar{w} = \Sigma w_i/n$.

(d) Calculate

$$W^2 = \sum_{i=1}^{n} \left\{w_i - \left(\frac{2i-1}{2n}\right)\right\}^2 + \frac{1}{12n} \; ; \; U^2 = W^2 - n(\bar{w} - \tfrac{1}{2})^2 \; ;$$

or

$$A^2 = -\frac{1}{n} \sum_{i=1}^{n} (2i-1) \{\ln w_i + \ln(1-w_{n+1-i})\} - n \; .$$

(e) Table 1 gives asymptotic percentage points for the test statistics. In order to use these with a sample of size n, we calculate <u>modified statistics</u> W^*, U^* and A^*, as follows. For $m = 1$, calculate $W^* = W^2(1+0.16/n)$, $U^* = U^2(1+0.16/n)$ or $A^* = A^2(1+0.6/n)$; for $m > 1$, calculate

$$W^* = \frac{(1.8nW^2 - 0.14)}{1.8n-1}$$

$$U^* = \frac{(1.8nU^2 - 0.14)}{1.8n-1}$$

$$A^* = A^2 + \frac{1}{n}\left(0.2 + \frac{0.3}{m}\right) \; .$$

The modified statistics are then referred to the upper tail asymptotic points to make the test; if a modified statistic exceeds the appropriate table entry in Table 1, for the statistic used and for given m and percentage level α, reject H_0 at significance level α.

<u>Illustration</u>. Suppose a sample of size $n = 10$ gives $W^2 = 0.170$, $U^2 = 0.1555$ and $A^2 = 1.102$, in a test for the gamma distribution with $m = 4$; from the formula above $W^* = (18(0.170) - 0.14)/17 = 0.172$;

similarly $U^* = (18(0.155) - 0.14/17 = 0.156$, and $A^* = 1.102 + 0.1(.275) = 1.130$. Comparison with the asymptotic points shows that W^* is significant at approximately the 5.8% level, U^* at the 5.3% level, and A^* at the 5.5% level.

For the different statistics, extensive Monte Carlo studies were made to give the percentage points for $n = 5,8,10,20$ and 50. These were plotted against $1/n$ and smoothed, and the modifications transform the percentage point for finite n to a value which closely approximates the asymptotic value for the same α. They have been devised to be independent of α, and, as much as possible, to be independent also of m; thus extensive tables of percentage points, for each combination of n, m and α, have been condensed into Table 1. If the true percentage level is α', the modifications give an error $|\alpha - \alpha'|$ less than 0.5% for $n \geq 5$, and for $n \geq 8$ the accuracy will usually be much higher.

3. APPLICATIONS.

3.1. Application to Tests for Multivariate Normality.

Suppose y is a vector of p observations, and it is wished to test H_0: a random sample $y_1, y_2, \ldots, y_n$ comes from a multivariate normal distribution with mean μ and covariance matrix Σ. The quadratic forms $r_i = (y_i - \mu)'\Sigma^{-1}(y_i - \mu)$, $i = 1,\ldots,n$ will, on H_0, be distributed as χ^2_p, and so can be tested to come from $G(r,m)$, where $m = p/2$. This idea, and others related to it, appears frequently in the literature. In such a case, all parameters are specified, and EDF statistics could be used with Case 0 (Stephens, 1974). However, it might be a more robust procedure to test instead for $G(\theta r,m)$, with

θ to be estimated; this concentrates on the shape of the quadratic form even if there is a misspecification of the parameters, particularly of Σ. Improvement in power for tests of distributional form, when an estimate of a parameter is used even though its value, on H_0, should be known, have already been noted by Stephens (1974) and Dyer (1974), in the context of tests of normality for univariate data.

When the μ and Σ of the quadratic forms r_i are replaced by their sample values $\bar{y}$ and $S = n^{-1} \Sigma(y_i-\bar{y})(y_i-\bar{y})'$. giving $\tilde{r}_i = (y_i-\bar{y})' S^{-1}(y_i-\bar{y})$, then each $\tilde{r}_i$ has an approximate marginal χ_p^2 distribution, see, for example, Gnanadesikan (1979, §5.4). The $\tilde{r}_i$ have the additional property that $\Sigma\tilde{r}_i = n$ so that the null distribution of the $\tilde{r}_i$'s is approximately the same as that of the z_i of section 2.2 and tests of H_0 can be made using the procedure of section 2.2 with $\tilde{r}_i$'s replacing z_i's, and $m = p/2$.

3.2. Time Series Applications.

A variation of Bartlett's test (see Cox and Lewis, 1966, §6.4) for the independence of intervals in a renewal process can be devised using the procedure of section 2.2. Bartlett's test involves the smoothing of m adjacent periodogram coordinates and then dividing by the spectral density function, which is assumed known up to some unknown scale factor. This gives rise to *scaled periodogram* values, $x_1,\ldots,x_n$, which have an approximate $G(\theta x,m)$ distribution function, when the correct spectral density function is used. The unknown scale parameter θ can be estimated using the technique of section 2.2 and the z_i's found. A test of the correctly specified spectral density function can be made using the procedure of section 2.2 for various values of m.

Another application follows ideas mentioned in Cox and Lewis (1966, §3.2). In a Poisson process, suppose we consider the times between the $(j-1)m$-th and jm-th events, $j = 1,2,\ldots$, for mixed m. Although it might not be reasonable to assume that the between m events times are independent and have the $G(\theta x,m)$ distribution function for say $m = 1$ or 2, it might be reasonable to believe that the assumption is true for $m = 4,5,\ldots$. A test of this hypothesis can be made by taking $x_1,x_2,\ldots$ as the times between the origin and m-th event, between the m-th event and the $2m$-th event, and so on, and then following section 2.2.

3.3. Survival Time Analysis.

In survival data analysis many models have been proposed for the distribution of survival time; see, for example, Kalbfleisch and Prentice (1980, §3). One model is to assume that the survival time t has a distribution so that $x = \exp(t)$ has the $G(\theta x,m)$ distribution function with θ unknown and perhaps m unknown. On the basis of a random sample of t's giving rise to $x_1,\ldots,x_n$, using the transformation $x = \exp(t)$, we can test the goodness-of-fit of the gamma model with θ estimated for various specified values of m. We might use the value of m giving rise to the least significant fit as an estimate of the true value of m. Note that the test procedures of section 2.2 cannot deal with censored observations.

3.4. Equality of Variances in ANOVA.

Suppose the variance of a random sample $y_1,\ldots,y_p$ is $s^2 = \Sigma(y_i-\bar{y})^2/(p-1)$. Given k such independent samples, each of size p, let s_i^2 be the variance of sample i, estimating the population variance σ_i^2. Assuming that the samples are normally distributed then the null hypothesis of homogeneity H_{01}: $\sigma_1^2 = \cdots = \sigma_k^2$ can be tested using the procedure of section 2.2

(a) Put the s_i^2 in ascending order.

(b) Calculate $z_i = (p-1)ks_i^2/(2\Sigma s_i^2)$.

(c) *Follow steps* (c,d,e) *in section* 2.1, *with* $m = (p-1)/2$, and with $n = k$.

This adaptation follows easily from the fact that $(p-1)s_i^2/(2\sigma^2)$ are, on H_{01}, independently Gamma distributed with $m = (p-1)/2$.

One disadvantage of using the EDF statistics, in this context, is that they suffer, but not nearly to the same extent as the traditional tests for homogeneity of variance (by these we mean Bartlett's, Hartley's and Cochran's tests, see Pearson and Hartley, 1966, pp. 202-4) from being biased, that is the power is less than the significance level, for alternatives which encompass both heterogeneity of variances (H_{01} not true) and non-normality of the samples $y_1,\ldots,y_p$. This happens when the y's come from short tailed distribtutions $(\beta_2 \leq 2)$ and the variances satisfy relationships such as $\sigma_1^2 = \cdots = \sigma_\ell^2$, $\sigma_{\ell+1}^2 = \cdots = \sigma_{2\ell}^2 = 2\sigma_1^2$, $\sigma_{2\ell+1}^2 = \cdots = \sigma_{3\ell}^2 = 3\sigma_1^2$, where ℓ is a divisor of k. These results for the EDF statistics and the traditional statistics were found using Monte Carlo methods, and were discussed in the earlier Technical Reports. However the results for Bartlett's statistic, M, can be deduced from Box's (1953) result which showed that if s^2 is a sample variance based on v degrees of freedom, from a population with Kurtosis β_2, then s^2 is distributed in large samples as a normal sample variance based on $v\delta$ degrees of freedom, where

$$\delta = 2(\beta_2 - 1)^{-1} ,$$

when $v \to \infty$ and k is kept constant. Box showed that, for homogeneous samples, Bartlett's M is asymptotically distributed as a $\delta^{-1} \chi^2_{k-1}$ random variable.

We therefore suggest that the joint normality of the samples is assessed using other goodness-of-fit procedures, such as those of Wilk and Shapiro (1968) or Pettitt (1977). If normality is accepted then H_{01} can be tested using the EDF statistics or Bartlett's statistic, which, of course, has the optimal property that it is equivalent to the likelihood ratio test of H_{01} against H_{01} not true.

3.5. *Test for the Dirichlet Distribution.*

Using the theory of Wilks (1962, §7.7.1) it follows that the $z_i/(mm)$ $(i=1,\ldots,n)$ behave like the order statistics of an $(n-1)$-variate Dirichlet random variable $\underset{\sim}{v}$ $(v_n = 1 - \Sigma_1^{n-1} v_i)$ with distribution $D(m,\ldots,m;m)$. Thus the procedure of section 2.2 can be used to test the goodness-of-fit of an observation on $\underset{\sim}{v}$ to the Dirichlet distribution.

The applications which have been discussed cover a range of uses of the Gamma distribution in statistical work. In most cases, a test of H_0 would probably only be one part of an overall analysis of the data, and would be complementary to other analyses and tests; previous experience with EDF tests suggests that they would be useful in this general context, and it is hoped to examine their efficacy in later work.

TABLE 1

UPPER TAIL PERCENTAGE POINTS OF THE

ASYMPTOTIC DISTRIBUTIONS OF W^2, U^2 and A^2.

Parameter m is the shape parameter in Equation (1). If the test is for a χ^2_p distribution, p = 2m.

Statistic	m	10%	5%	2.5%	1%
	1	.175	.222	.270	.338
	2	.156	.195	.234	.288
	3	.149	.185	.222	.271
	4	.146	.180	.215	.262
	5	.144	.177	.211	.257
W^2	6	.142	.175	.209	.254
	8	.140	.173	.205	.250
	10	.139	.171	.204	.247
	12	.138	.170	.202	.245
	15	.138	.169	.201	.244
	20	.137	.169	.200	.243
	∞	.135	.165	.196	.237
	1	.129	.159	.189	.230
	2	.129	.158	.188	.228
	3	.128	.158	.187	.227
	4	.128	.158	.187	.227
	5	.128	.158	.187	.227
	6	.128	.157	.187	.227
U^2	8	.128	.157	.187	.227
	10	.128	.157	.187	.227
	12	.128	.157	.187	.227
	15	.128	.157	.187	.227
	20	.128	.157	.187	.227
	∞	.128	.157	.187	.227
	1	1.062	1.321	1.591	1.959
	2	.989	1.213	1.441	1.751
A^2	3	.959	1.172	1.389	1.683
	4	.944	1.151	1.362	1.648
	5	.935	1.139	1.346	1.627
	6	.928	1.130	1.335	1.612
	8	.919	1.120	1.322	1.595
	10	.915	1.113	1.314	1.583
	12	.911	1.110	1.310	1.578
	15	.908	1.106	1.304	1.570
	20	.905	1.101	1.298	1.562
	∞	.893	1,087	1.281	1.551

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A.N. Pettitt
University of Technology,
Loughborough, Leics., England

and

M.A. Stephens
Department of Mathematics
Simon Fraser University
Burnaby, B.C. Canada V5A 1S6

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EDF STATISTICS FOR TESTING FOR THE GAMMA DISTRIBUTION, WITH APPLICATIONS

Tests are given for testing that a random sample of size n comes from a Gamma distribution, with unknown scale parameter. The statistics examined are the EDF statistics W^2, U^2, and A^2. Asymptotic points are given for the statistics, and modifications to enable them to be used for finite n. Various applications of the test procedures are discussed.